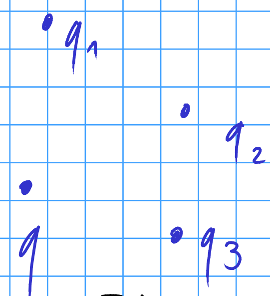
EM Tutorübung, Blatt 3, 17.05.2010

→ Elektrostatik: keinerlei bewegte Ladung ("kein Stromfluss")

Coulomb-Gesetz: $\vec{F}_{el} = \frac{q}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{\|\vec{r}-\vec{r}_i\|^3} (\vec{r}-\vec{r}_i)$

$$F_{el} = \frac{q^2}{4\pi\varepsilon r^2}$$



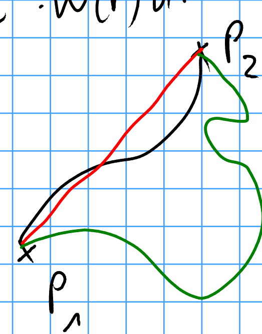

$$\boxed{\vec{E} = \frac{\vec{F}_{el}}{q}} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{\|\vec{r}-\vec{r}_i\|^3} (\vec{r}-\vec{r}_i)$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|}$$

Elektrostatik: $\vec{E} = -\nabla\Phi$

$\vec{E}$: Gradientfeld: $\operatorname{rot}\vec{E} \equiv 0 \Leftarrow \operatorname{rot}(-\nabla\Phi) = -\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \partial_{x_3}\end{pmatrix} \times \left(\begin{pmatrix}\partial_{x_1}\phi\\ \partial_{x_2}\phi\\ \partial_{x_3}\phi\end{pmatrix}\right)$

$$U = \int_W \vec{E}\, d\vec{r} = \int_{w(a)}^{w(b)} \vec{E}^T \dot{w}(t)\, dt \quad \int_{P_1}^{P_2} \vec{E}^T \cdot \dot{w}(t)\, dt$$




$w: D \subset \mathbb{R} \to \mathbb{R}^3$

$D = [a, b]$

⇒ Wegunabhängig

Exkurs: $F(x,y)$ ist konservativ, falls gilt $\partial_y F_x = \partial_x F_y$

$$F(x,y) = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix}$$

## Taylorreihe:

$$y = m(x - x_0) + y_0$$

$$f(x+h) = f(x) + \frac{f'(x)}{1!} h + \frac{f''(x)}{2!} h^2 + \dots + \frac{f^{(n)}}{n!} h^n$$

$$\underset{n \to \infty}{=} \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n}$$

Taylorreihe

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Bsp:

| | |
|---|---|
| $f(x) = \sin x$ | $f(0) = 0$ |
| $f'(x) = \cos x$ | $f'(0) = 1$ |
| $f''(x) = -\sin x$ | $f''(0) = 0$ |
| $f'''(x) = -\cos x$ | $f'''(0) = -1$ |
| $f^{(4)}(x) = \sin x$ | $f^{(4)}(0) = 0$ |
| $f^{(5)}(x) = \cos x$ | $f^{(5)}(0) = 1$ |

$$f(x+h) = f(h) = h - \frac{h^3}{6} + \frac{h^5}{120} + \mathcal{O}(h^6)$$ } vgl. Plot am Ende

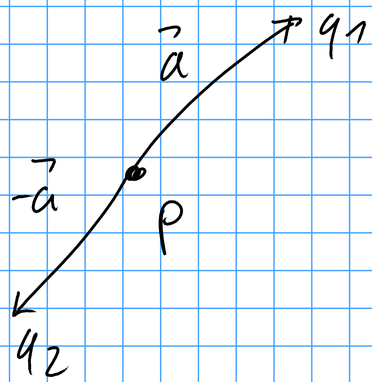
Mehrdimensionaler Fall: ($f$ Skalarfeld: $\mathbb{R}^n \to \mathbb{R}$)

$$f(x+h) = f(x) + \nabla f(x)\, h + \mathcal{O}(h^2)$$

$x \in \mathbb{R}^n$

$h \in \mathbb{R}^n$

## Aufgabe 9




$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\left[\frac{q_1}{\|\vec{r}-\vec{r}_1\|} + \frac{q_2}{\|\vec{r}-\vec{r}_2\|}\right] =$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{q_1}{\|\vec{r}-\vec{a}\|} + \frac{q_2}{\|\vec{r}+\vec{a}\|}\right]$$

(a) $q_1 = q_2 = q$

$$\Rightarrow \phi(\vec{r}) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\|\vec{r}-\vec{a}\|} + \frac{1}{\|\vec{r}+\vec{a}\|}\right]$$

$$f(\vec{r}) = \frac{1}{\|\vec{r}\|} = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$\nabla f(\vec{r}) = \begin{pmatrix} \partial x_1 \dfrac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \partial x_2 \dfrac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \partial x_3 \; \dots \end{pmatrix} = \begin{pmatrix} \dfrac{-\dfrac{1}{2\sqrt{x_1^2+x_2^2+x_3^2}}\cdot 2x_1}{(x_1^2+x_2^2+x_3^2)} \\ \vdots \\ \vdots \end{pmatrix}$$

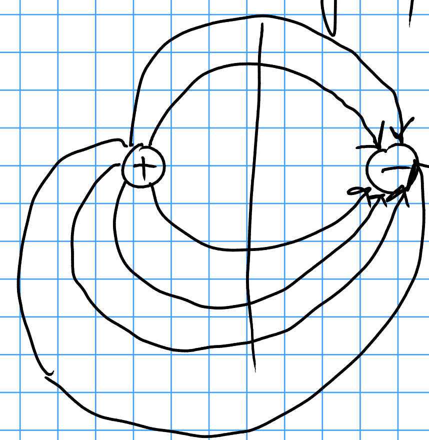
$$= \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \cdot \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}^{\,3}} = -\frac{\vec{r}}{\|\vec{r}\|^3}$$

richtig

$$\Phi(\vec r) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\|\vec r-\vec a\|}+\frac{1}{\|\vec r+\vec a\|}\right] =$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{1}{\|\vec r\|}+\frac{\vec r\cdot\vec a}{\|\vec r\|^3}+\frac{1}{\|\vec r\|}-\frac{\vec r\cdot\vec a}{\|\vec r\|^3}\right] = \frac{2q}{4\pi\varepsilon\|\vec r\|}$$

Interpretation: Punktladung im Zentrum mit Ladung $2q$.




b) $q_1 = -q_2$

$$\Phi(\vec r) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\|\vec r-\vec a\|}-\frac{1}{\|\vec r+\vec a\|}\right] =$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{1}{\|\vec r\|}+\frac{\vec r\cdot\vec a}{\|\vec r\|^3}-\left(\frac{1}{\|\vec r\|}-\frac{\vec r\cdot\vec a}{\|\vec r\|^3}\right)\right] =$$

$$= \frac{2q}{4\pi\varepsilon}\frac{\vec r\cdot\vec a}{\|\vec r\|^3}$$

$\Rightarrow$ Dipolfeld

## 8. Aufgabe

$$\vec x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$w(t) = R\,\vec e_r(\varphi) + z\,\vec e_z \qquad \vec e_r = \cos(2\pi t)\,\vec e_x + \sin(2\pi t)\,\vec e_y$$

$$= R\,\vec e_r(\varphi) + \frac{h}{3}t\,\vec e_z$$

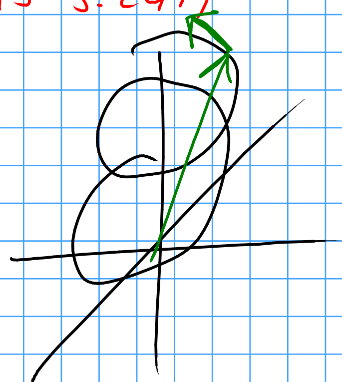
$$L = \int_w f \, ds \underset{f=1}{=} \int_a^b \|\dot{w}(t)\| \, dt$$

$$\frac{dw(t)}{dt} = R\left(-2\pi \sin(2\pi t)\vec{e}_x + 2\pi\cos(2\pi t)\vec{e}_y\right) + \frac{h}{3}\vec{e}_z$$

$$= 2\pi R \underbrace{\left(-\sin(2\pi t)\,\vec{e}_x + \cos(2\pi t)\vec{e}_y\right)}_{\vec{e}_\varphi} + \frac{h}{3}\vec{e}_z =$$

(Springer-FS S. 247)

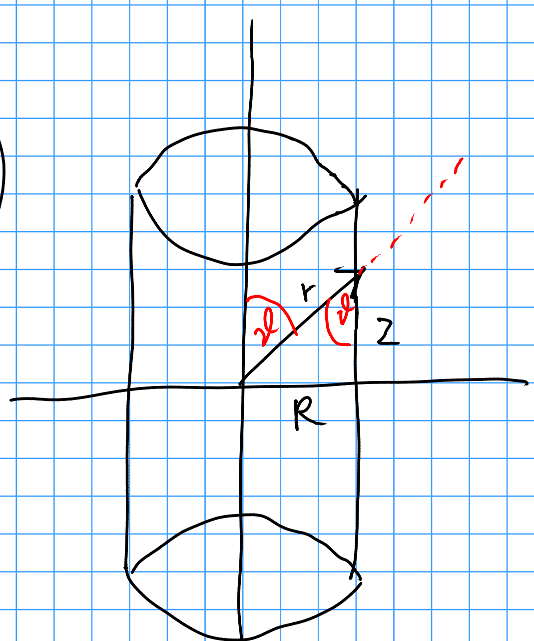
$$= 2\pi R \vec{e}_\varphi + \frac{h}{3}\vec{e}_z$$



$$\left\| \frac{dw(t)}{dt} \right\| = \sqrt{(2\pi R)^2 + \left(\frac{h}{3}\right)^2}$$

$$L = \int_0^3 \sqrt{(2\pi R)^2 + \left(\frac{h}{3}\right)^2} \, dt = \sqrt{(2\pi R)^2 + \left(\frac{h}{3}\right)^2} \cdot 3 \approx 151{,}3\,m$$

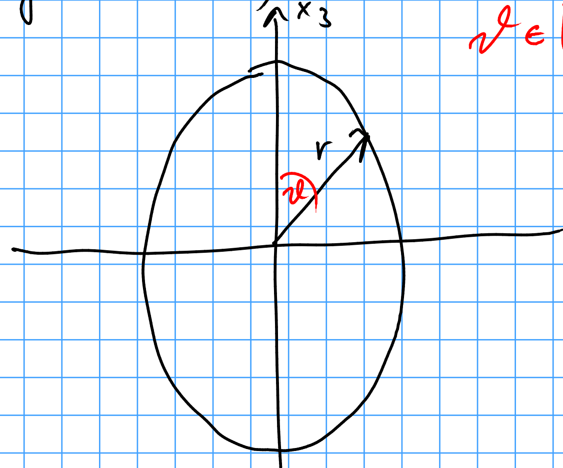
b)




$$\vec{x}(r, \vartheta, \varphi) = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

(Springer S. 248)

$\vartheta \in [0; \pi]$

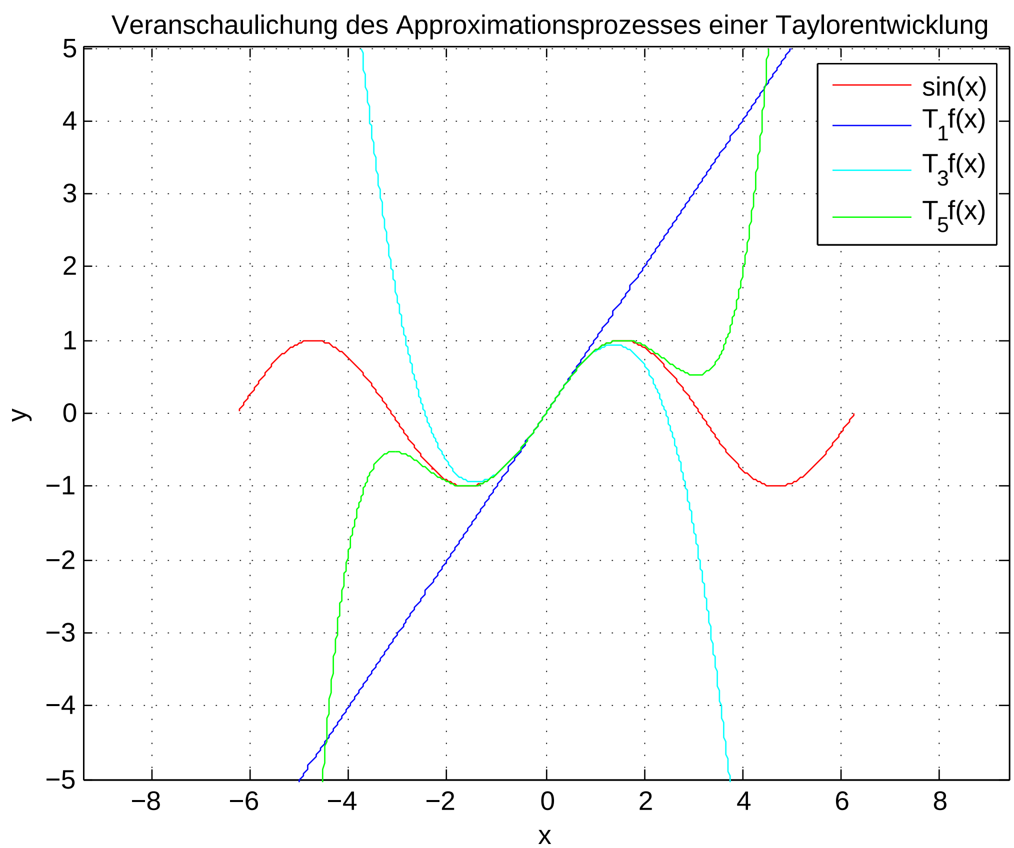


$$\tan\vartheta = \frac{R}{z} \Rightarrow \vartheta = \arctan\left(\frac{R}{z}\right)$$

$$r = \sqrt{R^2 + z^2} = \sqrt{R^2 + \left(\frac{h}{3}t\right)^2}$$

Parametrisierung in Kugelkoordinaten:

$$\vec{x}(r, \vartheta, \varphi) = r\,\vec{e}_r(\vartheta, \varphi) = \sqrt{R^2 + \left(\frac{h}{3}\right)^2}\,\vec{e}_r(\vartheta, \varphi)$$


Veranschaulichung des Approximationsprozesses einer Taylorentwicklung
sin(x)
T1f(x)
T3f(x)
T5f(x)
y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−8
−6
−4
−2
0
2
4
6
8